

Order-chaos transitions in field theories with topological terms: a dynamical systems approach

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Abstract

We present a comparative study of the dynamical behaviour of topological systems of recent interest, namely the non-Abelian Chern-Simons Higgs system and the Yang-Mills Chern-Simons Higgs system. By reducing the full field theories to temporal differential systems using the assumption of spatially homogeneous fields, we study the Lyapunov exponents for two types of initial conditions. We also examine in minute detail the behaviour of the Lyapunov spectra as a function of the various coupling parameters in the system. We compare and contrast our results with those for Abelian Higgs, Yang-Mills Higgs and Yang-Mills Chern-Simons systems which have been discussed by other authors recently. The role of the various terms in the Hamiltonians for such systems in determining the order-disorder transitions is emphasized and shown to be counter-intuitive in the Yang-Mills Chern-Simons Higgs systems.

1 Introduction

In recent times, the theory of dynamical systems has provided much insight into the origin of chaos in classical systems which were traditionally thought of as being completely deterministic. However, much of the progress has been mainly in the context of discrete mapping and those differential dynamical systems having low dimensional phase spaces.

In the context of mathematical physics , there exist differential dynamical systems described by a large number of variables and therefore having phase spaces of rather large dimensions. Examples of such systems are the Yang-Mills system (YM), the Chern-Simons system (CS) and their various enlargements such as the Yang-Mills Higgs (YMH) , Yang-Mills Chern-Simons Higgs (YMCSH) and Chern-Simons Higgs (CSH) systems. These systems are treated as dynamical systems after being derived from the original highly nonlinear partial differential equations (PDE'S) through the assumption of spatial homogeneity which reduces the dependence of the dynamical variables on the three or four-dimensional space-time coordinates to a dependence purely on time. The systems then become temporal differential dynamical systems.

Savvidy et.al [1] were the the first to demonstrate the chaotic nature of gaugetheories by treating them as differential dynamical systems. In fact, a detailed investigation has been conducted in this context [2] to classify the dynamical version of the pure YM field theory in terms of its ergodicity properties.

While the condition for ergodic behaviour of a system is that the generic phase trajectory visits all regions of phase space given a sufficiently long time, and all phase-averages can be replaced by time-averages, the YM system exhibits stronger stochasticity properties in phase space. Investigations reveal that it is definitely a mixing system, i.e., no time-averaging is required to achieve 'equilibrium'. In contrast to systems which are simply ergodic its spectrum is continuous. Indeed, the studies seem to indicate that the YM dynamical system is a Kolmogorov K-system, which exhibits stronger mixing properties than those mentioned above. A connected neighbourhood of phase trajectories in this case, exhibits a positive average rate of exponential divergence or net positive Lyapunov

exponent. Equivalently, by a remarkable theorem [3] a K system has positive Kolmogorov-Sinai (KS) entropy which is a measure of the degree of chaos, analogous to entropy as a measure of disorder in statistical mechanics. In fact, while the KS entropy itself is not straightforward to measure, the fact that it is a sum of the positive Lyapunov exponents implies that it is an important concept in the classification of chaotic systems.

Another aspect to the study of gauge theories as dynamical systems is the attempt to understand the ground (vacuum) state structure of quantum chromodynamics (QCD), and also its behaviour in extreme environments (such as high temperature). In this context, systems such as YMH, YMCS and YMCSH have been studied [4].

The addition of the CS term to various Abelian and non-Abelian gauge theories leads to novel features in general, as it is a topological term. Even in the complete PDE's, while a study of the YM case reveals interesting results in connection with the geometry and topology of four-dimensional manifolds, the related CS PDE's have yielded information about three-dimensional manifolds. The symplectic structure of CS theories differs in important ways from that of the Maxwell or YM gauge theories. In perturbative gauge theories the pure CS theories exhibit features that are absent in the Abelian gauge theory with both the Maxwell and CS terms. Delicate aspects relating to the infrared and ultraviolet behaviour and the regularization dependence in such perturbative theories [5], the natural connection of quantized three-dimensional CS gauge theories with two-dimensional conformal field theories [6], and the effect of the YM term which acts as a singular perturbation when added to the $SU(2)$ CS theory [7], have all been extensively explored in the literature.

Another aspect of interest in CS theories relates to its quantum mechanics. Non-perturbative quantum mechanical anomalies in these theories [8], infinite dimensional symmetry groups that arise in CS quantum mechanics [9], self-dual CS theories and extended supersymmetry [10] are a few areas in which distinct signatures of CS theories in sharp contrast to those of other gauge theories have been reported.

In this paper we report on yet another aspect of CS theories and contrast them with

other gauge theories. This pertains to the chaotic nature of gauge theories mentioned earlier. For our purpose, the equations of motion are made to evolve only temporally, by suppressing the spatial dependence. While the resulting equations are like the continuum analogues of discrete maps (the latter being an area where extensive work has been done on their chaotic behaviour), they are also the dynamical version of the full gauge theory and hence represent one sector of the corresponding field theory. In the literature this is used as a convenient reduction to examine the integrability properties of the theory. This is because if this sector is proven to be non-integrable the corresponding field theory also will be chaotic [11].

In our recent papers, we have shown that the Abelian CSH system without a kinetic term is integrable, while the inclusion of a kinetic term, making it into a Maxwell Chern-Simons Higgs (MCSH) system (or Yang-Mills with a $U(1)$ symmetry group) yielded a non-integrable system which admitted chaos [12]. The systems were also examined for the Painlevé property. A numerical study of the Lyapunov exponents and phase space trajectories were carried out to show the existence of chaos in these systems. In [13], the analysis was extended to the non-Abelian CSH and YMCSH system with an $SU(2)$ symmetry group and both were found to be chaotic.

In all these studies, one aspect that deserves more attention is the possibility of observing the existence of a phase transition, i.e., is there a sharp order to chaos transition in the parameter space of these theories? Of course in these systems energy can also be used as a parameter since it is a conserved quantity. Indeed in ref.[14], it has been claimed that an order-chaos transition is seen within a narrow range of energies. More recently, Kawabe [15] has argued that the Abelian Higgs system (YMH with a $U(1)$ symmetry group) also shows an order-chaos transition for certain ratios of the two parameters in the theory. Our paper examines this aspect of chaos in YM systems by studying the non-Abelian CSH (NACSH) and the YMCSH systems with an $SU(2)$ symmetry group. A comparative study is done to see the role of the kinetic term, the Higgs term and the CS term in the transition.

Some interesting features regarding the details of the phase transition from order to chaos in the dynamical analogues of both Abelian and non-Abelian gauge theories have been reported in the literature. In the context of Abelian Higgs theories Kawabe has reported transition from order to chaos within a certain range of the Higgs coupling constant and energy [15]. The onset of chaos is remarkably different qualitatively from the corresponding transition in the YMCS system where Giansanti and Simic [14] have reported the existence of an interesting ‘fractal’ structure in the phase transition region. Much earlier Savvidy et. al [16] have observed that the role of the Higgs is to order the Yang-Mills system and later extensive work on the YMH systems was conducted in ref [17]. The picture that emerges therefore, is that the Higgs and the Chern-Simons term have distinct and different roles to play in the transition. It is therefore of importance to examine the effect of both terms on the Yang -Mills field in the YMCSH theory. As a primer to this, the competing effect of the ‘oscillatory’ behaviour of the CS term and the ‘stabilizing’ role of the Higgs term in the CSH system is investigated in this paper. Later, we contrast this with the corresponding results in the YMCSH system. It is thus obvious that a K system like the Yang-Mills theory when coupled to CS and Higgs must show a rich and instructive behaviour in the understanding of regularity vs. chaos in Hamiltonian systems. A second aspect which emerges is related to an interesting question - will the Higgs stabilize any gauge invariant term involving only the vector potentials, independent of whether it is of the YM type or the CS type or is it necessary that the gauge field is only YM in nature? We attempt to answer these questions in this paper.

2 The Dynamical Systems

In this section we set up the two systems we shall be examining. Let us consider first the non-Abelian pure Chern-Simons Higgs system (i.e., without the kinetic term). From our earlier paper [13], the Lagrangian for the non-Abelian (SU(2)) CSH (NACSH) system in 2+1 dimensions in Minkowski space is given by:

$$L = \frac{m}{2} \epsilon^{\mu\nu\lambda} [F_{\mu\nu}^a A_\lambda^a - \frac{g}{3} f_{abc} A_\mu^a A_\nu^b A_\lambda^c] + D_\mu \phi_a^\dagger D^\mu \phi_a - V(\phi) \quad (1)$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc}A_\mu^b A_\nu^c, \quad (2)$$

f_{abc} are the structure constants of the SU(2) Lie algebra and

$$D_\mu \phi_a = (\partial_\mu - igT^l A_\mu^l) \phi_a. \quad (3)$$

Here, T_a are the generators of the SU(2) algebra, so that $\text{tr}[T_a T_b] = \delta_{ab}$. The equations of motion become:

$$m\epsilon^{\nu\alpha\beta} F_{\alpha\beta}^a = ig[D^\nu \phi^\dagger T_a \phi - \phi^\dagger T_a D^\nu \phi] \quad (4)$$

$$D_\mu D^\mu(\phi) = -\frac{1}{2} \frac{\partial V}{\partial \phi} \quad (5)$$

Then, considering the real triplet representation for the Higgs field and the spatially homogeneous solutions $\partial_i A_j^a = \partial_i \phi = 0$ i,j=1,2 and the gauge choice $A_0^a = 0$ we get for the $\nu = 0$ component of eqn.[4]

$$m\vec{A}_1 \times \vec{A}_2 = -\vec{\phi} \times \vec{\phi} \quad (6)$$

which is just the Gauss' law constraint. The remaining equations of motion for the vector field are:

$$\dot{\vec{A}}_1 = \frac{g^2}{m}(\vec{A}_2 \vec{\phi}^2 - \vec{\phi} \vec{A}_2 \cdot \vec{\phi}) \quad (7)$$

$$\dot{\vec{A}}_2 = -\frac{g^2}{m}(\vec{A}_1 \vec{\phi}^2 - \vec{\phi} \vec{A}_1 \cdot \vec{\phi}) \quad (8)$$

The equation of motion for the Higgs field is:

$$\ddot{\vec{\phi}} = -g^2[(\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi} - (\vec{A}_1 \cdot \vec{\phi} \vec{A}_1 + \vec{A}_2 \cdot \vec{\phi} \vec{A}_2)] - \frac{1}{2} \frac{\partial V}{\partial \vec{\phi}}. \quad (9)$$

From the equations of motion for the vector fields it is easily seen that $\vec{A}_1^2 + \vec{A}_2^2$ is a constant of the motion. Throughout this paper, we work with the potential

$$V(\phi) = \frac{\lambda}{4}(\vec{\phi}^2 - v^2)^2. \quad (10)$$

The NACSH system described by the above equations of motion has three parameters. For comparison with the work of Kawabe [15], we shall scale the variables such that we

are left with only one parameter. The following scaling of variables,

$$\vec{A}_1 \longrightarrow g\vec{A}_1$$

$$\vec{A}_2 \longrightarrow g\vec{A}_2$$

$$\vec{\phi} \longrightarrow \frac{g}{\sqrt{m}}\vec{\phi}$$

$$v \longrightarrow \frac{g}{\sqrt{m}}v$$

reduces the equations of motion to:

$$\vec{A}_1 = [\vec{A}_2\vec{\phi}^2 - \vec{\phi}(\vec{A}_2 \cdot \vec{\phi})] \quad (11)$$

$$\vec{A}_2 = -[\vec{A}_1\vec{\phi}^2 - \vec{\phi}(\vec{A}_1 \cdot \vec{\phi})] \quad (12)$$

$$\vec{\phi} = -[(\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi} - (\vec{A}_1 \cdot \vec{\phi}\vec{A}_1 + \vec{A}_2 \cdot \vec{\phi}\vec{A}_2)] - \frac{\kappa}{2}\vec{\phi}(\vec{\phi}^2 - v^2) \quad (13)$$

with $\kappa = \frac{\lambda m}{g^2}$. In the rest of the paper, we shall set the scaled v to be one without loss of generality. The scaled Lagrangian is given by

$$L = (\vec{A}_1 \cdot \vec{A}_2 - \vec{A}_2 \cdot \vec{A}_1) + \vec{\phi}^2 - [(\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi}^2 - (\vec{A}_1 \cdot \vec{\phi})^2 - (\vec{A}_2 \cdot \vec{\phi})^2] - \frac{\kappa}{4}(\vec{\phi}^2 - 1)^2 \quad (14)$$

The corresponding energy function is easily seen to be

$$E = \vec{\phi}^2 + [(\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi}^2 - (\vec{A}_1 \cdot \vec{\phi})^2 - (\vec{A}_2 \cdot \vec{\phi})^2] + \frac{\kappa}{4}(\vec{\phi}^2 - 1)^2. \quad (15)$$

These are the equations governing the NACSH dynamical system. Since we wish to compare the results for the NACSH system with those of the YMCSH system we proceed to set up the YMCSH dynamical equations. The Lagrangian is given by

$$\begin{aligned} L = & -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + \frac{m}{2}\epsilon^{\mu\nu\alpha}[F_{\mu\nu}^a A_\alpha^a - \frac{g}{3}f_{abc}A_\mu^a A_\nu^b A_\alpha^c] \\ & + D_\mu\phi_a^\dagger D^\mu\phi_a - V(\phi) \end{aligned} \quad (16)$$

The equations of motion are

$$D_\mu F^{\mu\nu a} + m\epsilon^{\nu\alpha\beta}F_{\alpha\beta}^a = ig[D^\nu\phi^\dagger T_a\phi - \phi^\dagger T_a D^\nu\phi] \quad (17)$$

$$D_\mu D^\mu\phi^a = -\frac{\partial V}{\partial\phi_a^*} = -\frac{1}{2}\frac{\partial V}{\partial\phi_a} \quad (18)$$

The $\nu = 0$ component gives the Gauss' law constraint which in this case is

$$\frac{1}{2}(\vec{A}_1 \times \dot{\vec{A}}_1 + \vec{A}_2 \times \dot{\vec{A}}_2 + 2m\vec{A}_1 \times \vec{A}_2) = -2\vec{\phi} \times \vec{\phi}. \quad (19)$$

Once again, we choose the gauge $A_0 = 0$ and consider the spatially homogeneous case ; then the equations of motion for the gauge fields become:

$$\begin{aligned} \ddot{\vec{A}}_1 + 2m\dot{\vec{A}}_2 + 2g^2(\vec{A}_1\vec{\phi}^2 - \vec{\phi}\vec{A}_1 \cdot \vec{\phi}) \\ + g^2(\vec{A}_1\vec{A}_2 \cdot \vec{A}_2 - \vec{A}_2\vec{A}_1 \cdot \vec{A}_2) = 0 \end{aligned} \quad (20)$$

$$\begin{aligned} \ddot{\vec{A}}_2 - 2m\dot{\vec{A}}_1 + 2g^2(\vec{A}_2\vec{\phi}^2 - \vec{\phi}\vec{A}_2 \cdot \vec{\phi}) \\ + g^2(\vec{A}_2\vec{A}_1 \cdot \vec{A}_1 - \vec{A}_1\vec{A}_2 \cdot \vec{A}_2) = 0. \end{aligned} \quad (21)$$

From these equations it is easy to see that

$$\vec{A}_2 \cdot \dot{\vec{A}}_1 - \vec{A}_1 \cdot \dot{\vec{A}}_2 + m(\vec{A}_1^2 + \vec{A}_2^2)$$

is a constant of the motion. The equation of motion for the three component Higgs field becomes

$$\ddot{\vec{\phi}} = -g^2[(\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi} - (\vec{A}_1 \cdot \vec{\phi}\vec{A}_1 + \vec{A}_2 \cdot \vec{\phi}\vec{A}_2)] - \frac{1}{2}\frac{\partial V}{\partial \vec{\phi}}. \quad (22)$$

In this case we have three second order differential equations for each vector field. The Lagrangian which leads to these equations of motion is:

$$\begin{aligned} L = \frac{1}{2}(\dot{\vec{A}}_1^2 + \dot{\vec{A}}_2^2) + m(\dot{\vec{A}}_1 \cdot \vec{A}_2 - \dot{\vec{A}}_2 \cdot \vec{A}_1) + \dot{\vec{\phi}}^2 \\ - g^2[\frac{1}{2}(\vec{A}_1^2\vec{A}_2^2 - (\vec{A}_1 \cdot \vec{A}_2)^2) + (\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi}^2 - (\vec{A}_1 \cdot \vec{\phi})^2 - (\vec{A}_2 \cdot \vec{\phi})^2] - V(\phi) \end{aligned} \quad (23)$$

Using the same rescaling as for NACSH we have the equations of motion

$$\begin{aligned} \frac{1}{m}\ddot{\vec{A}}_1 + 2\dot{\vec{A}}_2 + 2(\vec{A}_1\vec{\phi}^2 - \vec{\phi}\vec{A}_1 \cdot \vec{\phi}) \\ + \frac{1}{m}(\vec{A}_1\vec{A}_2 \cdot \vec{A}_2 - \vec{A}_2\vec{A}_1 \cdot \vec{A}_2) = 0 \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{1}{m}\ddot{\vec{A}}_2 - 2\dot{\vec{A}}_1 + 2(\vec{A}_2\vec{\phi}^2 - \vec{\phi}\vec{A}_2 \cdot \vec{\phi}) \\ + \frac{1}{m}(\vec{A}_2\vec{A}_1 \cdot \vec{A}_1 - \vec{A}_1\vec{A}_2 \cdot \vec{A}_2) = 0 \end{aligned} \quad (25)$$

and

$$\ddot{\vec{\phi}} = -[(\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi} - (\vec{A}_1 \cdot \vec{\phi}\vec{A}_1 + \vec{A}_2 \cdot \vec{\phi}\vec{A}_2)] - \frac{\kappa}{2}\vec{\phi}(\vec{\phi}^2 - 1). \quad (26)$$

It is interesting to note that while in the NACSH system the Yang-Mills parameter g , the Higgs parameter λ and the Chern-Simons parameter m could all be combined into the parameter κ , this is not possible for the YMCSH system where we are left with both κ and m appearing explicitly. The scaled Lagrangian in this case is given by

$$L = \frac{1}{2m}(\dot{\vec{A}}_1^2 + \dot{\vec{A}}_2^2) + (\dot{\vec{A}}_1 \cdot \vec{A}_2 - \dot{\vec{A}}_2 \cdot \vec{A}_1) + \dot{\vec{\phi}}^2 - \frac{1}{2m}[\vec{A}_1^2 \vec{A}_2^2 - (\vec{A}_1 \cdot \vec{A}_2)^2] + [(\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi}^2 - (\vec{A}_1 \cdot \vec{\phi})^2 - (\vec{A}_2 \cdot \vec{\phi})^2] - \frac{\kappa}{4}(\vec{\phi}^2 - 1)^2 \quad (27)$$

with an energy function

$$E = \frac{1}{2m}(\dot{\vec{A}}_1^2 + \dot{\vec{A}}_2^2) + \dot{\vec{\phi}}^2 + \frac{1}{2m}[\vec{A}_1^2 \vec{A}_2^2 - (\vec{A}_1 \cdot \vec{A}_2)^2] + [(\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi}^2 - (\vec{A}_1 \cdot \vec{\phi})^2 - (\vec{A}_2 \cdot \vec{\phi})^2] + \frac{\kappa}{4}(\vec{\phi}^2 - 1)^2. \quad (28)$$

This completes the description of the dynamical systems which we shall be studying. In the next section, we describe the numerical analysis that we have carried out.

3 Numerical Analysis

It is clear from the above dynamical equations that the phase space of these systems are large. One of the traditional ways for examining phase spaces to determine chaotic vs. regular behaviour has been to use the Poincaré sections where one examines the points mapped out on a plane surface in phase space as the trajectory crosses it. While this is straightforward for dynamical systems whose phase space dimensionality does not exceed four, it is difficult to interpret it in the systems we are dealing with.

Instead, we examine the variation of the maximal Lyapunov exponent as the two NACSH parameters energy and κ are varied. This clearly shows us regions of regular behaviour (where the exponent goes to zero) and regions of chaotic behaviour (where the exponent is positive). These calculations were carried out for a wide range of initial conditions. The initial conditions that were chosen were in turn dictated by the dynamical systems themselves. Being derived from the equations of motion the field variables are required to satisfy the Gauss' law constraint. Since this constraint must be preserved during the time evolution of the dynamical system, it is sufficient to ensure their validity via the initial conditions.

For the NACSH system the following forms were chosen as the initial conditions:

$$\vec{A}_1 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \quad \vec{A}_2 = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} \quad \vec{\phi} = \begin{pmatrix} -x \\ x \\ 0 \end{pmatrix} \quad \dot{\vec{\phi}} = \frac{1}{2} \begin{pmatrix} x \\ x \\ 0 \end{pmatrix}. \quad (29)$$

For YMCSH this is supplemented with

$$\dot{\vec{A}}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \dot{\vec{A}}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (30)$$

x was then varied to obtain a range of initial conditions for suitable energies of interest. In figs. 1–4 we show the behaviour of the maximal Lyapunov exponent as a function of energy for $\kappa = 0, .5, 1, 5$. For $\kappa = 0$ (i.e., absence of the Higgs potential) we see that the system is mostly **chaotic** with a window of regularity for $7 \leq E \leq 9$. Increasing the value of κ , we find a transitional region of regular to chaotic behaviour at small energies, in contrast to mostly oscillatory behaviour at higher energies. For $\kappa = 1$, more transitions from order to chaos appear at larger energies. Much more dramatic behaviour is seen for large κ ($\kappa = 5$), where oscillatory behaviour manifests itself for $E \geq 3$. Thus we see that the effect of the topological term (large m) is to produce a regular oscillatory behaviour in the dynamics of the system.

To illustrate the regular and chaotic behaviour on the trajectories of this system we exhibit in figs. 5 and 6, phase plots corresponding to the energies at which the Lyapunov exponent shows regular and chaotic behaviour. These correspond to $\kappa = 1$ and energies 10.28 and 20.607.

Giansanti and Simic [14] report a fractal-like structure of order-chaos transitions in YMCS systems. In our particular case, when the Yang- Mills field is absent and the Higgs field is present we see no such fractal behaviour in the region of phase space that we have examined. Therefore, this suggests that the quartic coupling that arises from the inclusion of the kinetic YM term may be responsible for the observed fractal structure. This rich phase space structure of the CSH, YMCS and YMCSH systems clearly needs further exploration. The set of trajectories examined by Giansanti and Simic do not correspond to the ansatz that we have chosen and therefore we have explored different regions of

phase space. Thus our results are complementary to those obtained by Giansanti and Simic.

We have examined the NACSH system for another ansatz which we may call the two-variable ansatz:

$$\vec{A}_1 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \quad \vec{A}_2 = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \quad \vec{\phi} = \begin{pmatrix} -x \\ y \\ 0 \end{pmatrix} \quad \vec{\phi} = \frac{1}{2} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}. \quad (31)$$

For YMCSH this is supplemented with

$$\vec{A}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{A}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (32)$$

This allows us to compare our results with that of Kawabe for the Abelian Higgs theory [15], where motion is completely bounded for $Q = \frac{4Eg^2}{\lambda} \leq 1$, while it is unbounded for $Q > 1$. In contrast, for the CSH system that we are studying, for all values of $Q = 4E/\kappa$ and $\kappa = \frac{\lambda}{mg^2}$, the motion is bounded. Indeed we find that for large E and large κ the contour levels of the function $W = [(\vec{A}_1^2 + \vec{A}_2^2)\vec{\phi}^2 - (\vec{A}_1 \cdot \vec{\phi})^2 - (\vec{A}_2 \cdot \vec{\phi})^2] + \frac{\kappa}{4}(\vec{\phi}^2 - 1)^2$ show extremely restricted domains for the dynamics. Fig. 7 represents the potential contours for $\kappa = 10$ for energies in the range $\{0.25, 2\}$ while fig. 8 gives the contours for the energy range $\{2, 10\}$. There is a dramatic change in the available phase space for this particular two-variable ansatz. This clearly suggests that the non-Abelian nature of the CS term is already contributing to a significant change in the dynamical structure.

Evidence for the transition from regularity to chaos is seen in fig. 9, where we show the fraction of the phase space that is regular. This is obtained by calculating the maximal Lyapunov exponent for $\kappa = 1$ for various initial conditions for energies ranging from 1 to 10. The cutoff on the exponent for regularity was taken to be 0.01. A simple count on the exponents falling below this value out of a hundred initial conditions for each energy was carried out. The figure clearly reveals that for small κ , the NACSH system becomes chaotic more or less monotonically as the energy is increased. However for large κ there is no such simple behaviour. A fuller discussion of this is given in the next section.

This transition between regular and chaotic behaviour is also seen in the phase space trajectories shown in figs. 10 and 11. In fig. 10 we plot ϕ_2 vs. ϕ_1 for $\kappa = 1$, energy=10

and an initial condition where the maximal Lyapunov exponent is almost zero. In fig. 11 we show the phase plot for the same parameter values for an initial condition which gives a large maximal Lyapunov exponent.

We now proceed to the numerical investigation of the YMCSH systems. Fig. 12 shows the variation of the maximal Lyapunov exponent as a function of energy for $\kappa = 1, 10$ and energy=10, with $m=1$. The graphs clearly show that for large κ (where either the Higgs coupling λ is large or the YM coupling g is small) the system exhibits more regularity for low energies.

This is to be contrasted with earlier work [1],[15], where it was reported that in a YMH system as λ increases the system becomes more regular regardless of the energy regime being investigated. Here, we see that for the YMCSH system such transitions to regularity are seen for small energies as κ increases while the Lyapunov exponent increases almost linearly with energy for the large energy regime. While fig.12 was obtained for the one-variable ansatz, fig.13 shows the maximal Lyapunov exponent as a function of the initial variable x for the two-variable ansatz, for fixed values of energy and κ . Once again we see regions of regularity for small energies and chaotic behaviour for large energies irrespective of the value of κ . Evidence for regularity and chaos for the two-variable ansatz is given in figs.13 and 14. In fig.14 we see that for $\kappa = 5$ and energy=1, the phase space trajectory is highly regular and quasi-periodic. Fig.15 shows a region for the same parameter values, where the phase space is chaotic.

All the calculations for the NACSH system were carried out using a straightforward Runge-Kutta fourth order routine with care being taken to preserve the constants of motion to an accuracy of one part in 10^5 . However, the calculations for the YMCSH system required an adaptive step-size Runge-Kutta routine to ensure energy conservation to the same degree of accuracy as in the NACSH system.

4 Results and Discussion

From the numerical studies undertaken by us, certain very interesting features emerge, not only regarding the richness of the phase space corresponding to gauge theories, but also with respect to some ‘counter-intuitive’ phenomena that occur in these systems.

One feature that seems to be common to both the YMH [16] and CSH systems, from an investigation of the potential contours corresponding to various initial conditions, is the boundedness of phase space. This is in contrast to the situation that prevails in the Abelian Higgs system [15]. The boundedness observed is better understood when we realize that the YMCS dynamical system can be used to describe particle motion in a magnetic field [14], a physical situation which allows only for bounded motion.

Apart from the crucial role played by the ansatz chosen by us, in this matter, the non-Abelian nature of the gauge term also plays an important part. In this context, we realize that the boundedness observed is quite independent of the details of the non-Abelian coupling, i.e., whether it is of the YM type or of the trilinear CS type.

We recall that the YM system is a K-system [2] and that the CS term creates more regular windows when added to the YM system [14]. This accounts to a large measure, for the fact that for some κ and m values, the maximal Lyapunov exponents in the YMCSH case are larger than the corresponding ones for the CSH system.

However, a striking feature that emerges in our studies that is counter-intuitive is that, in general, it is not true that increase in κ ‘regularizes’ the gauge term at all energies. Recall that our calculations are made in terms of the scaled parameter κ . An increase in κ could either be due to an increase in the Higgs coupling λ or a decrease in the gauge coupling g , for a fixed m . Whereas in the former case the regularity is expected to increase, in the latter case, it is not established that for all small non-zero g more regular islands appear. In fact, as borne out by figs. 12,13 and 14, **an increase in κ produces more regularity only for small values of energy**. This can be understood better if we realize that it is not just the value of κ that determines the appearance of regular islands, but also the available phase space as well. This is clear from the energy contours shown

in figs.7 and 8. With more accessible regions in phase space, it is but natural that more randomness sets in and the KAM tori structure gets broken.

Another aspect is that as the energy increases, the maximal Lyapunov exponent increases in magnitude in the YMCSH system. This shows that the YM terms takes over for large energies and the CS term produces the ‘oscillatory effect’. The effect of the CS term is reminiscent of the ‘fractal’ structure observed in YMCS systems, where in various energy windows, order-chaos-order transitions are observed [14].

An important feature that we see is that while in the pure CSH system large windows of regularity exist (as seen for the range of energies shown in figs.1-4), in YMCSH this does not happen. This perhaps can be traced back to the interplay between the oscillatory effect of the CS term, the regularizing effect of the Higgs term and the completely chaotic nature of the YM term. Further , with increase in κ , transitions from regularity to chaos take place in the pure CSH case at relatively higher energies when compared with the low energy values corresponding to order-chaos transitions in YMCSH.

The final picture which emerges bears out the fact that in a complex dynamical system with a large phase space (in contrast to the wide class of Hamiltonian systems with two degrees of freedom) curious interplay between different coupling constants and the rich structure of phase space itself can lead to novel results- some of them quite counter-intuitive and surprising. A detailed examination of dynamical systems which emerge from field theoretic systems are thus vital for an understanding of Hamiltonian systems with a large number of degrees of freedom. In turn it is also an important primer in the understanding of non-Abelian field theories themselves.

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6 Figure Captions

Fig.1 Variation of the maximal Lyapunov exponent with energy for NACSH for $\kappa = 0$.

Fig.2 Variation of the maximal Lyapunov exponent with energy for NACSH for $\kappa = 0.5$.

Fig.3 Variation of the maximal Lyapunov exponent with energy for NACSH for $\kappa = 1.0$.

Fig.4 Variation of the maximal Lyapunov exponent with energy for NACSH for $\kappa = 5.0$.

Fig.5 Phase plot of ϕ_2 vs. ϕ_1 showing regular behaviour for NACSH for the one-variable ansatz for $\kappa = 1$, $E=10.2088$ and $x=1.381$.

Fig.6 Phase plot of ϕ_2 vs. ϕ_1 showing chaotic behaviour for NACSH for the one-variable ansatz for $\kappa = 1$, $E=20.6071$ and $x=1.64$.

Fig.7 Contour plot for the potential corresponding to NACSH for $\kappa = 10$ and energy range $\{.25, 2\}$

Fig.8 Contour plot for the potential corresponding to NACSH for $\kappa = 10$ and energy range $\{2, 10\}$

Fig.9 Phase space fraction of regularity vs. energy for $\kappa = 1$ for NACSH

Fig.10 Phase plot of ϕ_2 vs. ϕ_1 showing regular behaviour for NACSH for the two-variable ansatz for $\kappa = 1$, $E=10$, $x=.05$ and $y=2.598405$

Fig.11 Phase plot of ϕ_2 vs. ϕ_1 showing chaotic behaviour for NACSH for the two-variable ansatz for $\kappa = 1$, $E=10$, $x=.9$ and $y=1.908953$.

Fig.12 Comparison of maximal Lyapunov exponent vs. energy for YMCSH for the one-variable ansatz for $\kappa = 1$, $m=1$ (* represents the curve) with $\kappa = 10$, $m=1$ (+ represents the curve).

Fig.13 Comparison of maximal Lyapunov exponent vs. x for YMCSH for the two-variable ansatz for $\kappa = 5$, $E=1, m=1$ (o represents the curve) with $\kappa = 10, E=5, m=1$ (+ represents the curve) and $\kappa = 1, E=10, m=1$ (* represents the curve)

Fig.14 Phase plot of ϕ_2 vs. ϕ_1 showing regular behaviour for YMCSH for the two-variable ansatz for $\kappa = 5$, $E=1$, $x=.725$ and $y=0.742085$.

Fig.15 Phase plot of ϕ_2 vs. ϕ_1 showing chaotic behaviour for YMCSH for the two-variable ansatz for $\kappa = 5$, $E=1$, $x=.9125$ and $y=0.571046$.

